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STABLE OUTCOMES FOR MULTI-COMMODITY FLOW GAMES

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ABSTRACT

Multi-commodity flow situations and flow games are considered with convex arc capacity sets and with a unique source and a unique sink for all commodities. It is shown that for all flow games of this type there exist stable outcomes. For two classes of flow situations, namely separable and uniform flow situations, it is proved that the bottle-neck set coincides with the flow value set.

1. INTRODUCTION

The theory of single-commodity flows in networks started with the pioneering work of Ford and Fulkerson [2]. The (im)possibility to extend their main result, the max-flow min-cut theorem, to the case of multi-commodity flows has been discussed by many researchers (cf. [1], [3]-[5], [8]-[11]).

One of the purposes of this paper is to give extensions of the Ford-Fulkerson theorem for certain classes of multi-commodity flow situations (MCF-situations).

The main purpose is to consider controlled multi-commodity flow situations (CMCF-situations) where the arcs in the network are possessed by different owners, and to tackle the question whether there exist stable outcomes for the corresponding game. The obtained theorem extends the result of Kalai and Zemel [6] of the non-emptyness of the core for flow games, arising from controlled single-commodity flow situations (CSCF-situations).

We will consider MCF-situations on directed networks where

(i) there is a unique source and a unique sink for all commodities,

(ii) the capacity set of each arc is a comprehensive, compact and convex subset of the commodity space.

Such situations will be called, in view of (ii), convex MCF-situations.

We recall that in many MCF-papers one presupposes:

(i)* there is for each commodity a source and a sink,

(ii)** the capacity set of each arc ℓ is described by a number $c(\ell)$, denoting that per unit of time only those commodity bundles can pass that arc, for which the sum of the amounts of each commodity does not exceed $c(\ell)$.

Although (i) is stronger than (i)*, in many practical situations condition (i) is valid. Condition (ii) is weaker and also more realistic than condition (ii)*, because in all kind of material flow problems and also in communication problems 'contraction effects' can occur i.e. one can often increase the flow possibilities through arcs, by mixing the commodities in an appropriate manner.

The organization of the paper is as follows. In section 2 a formal description of the models and the necessary definitions are given. Section 3 is devoted to extensions of the max-flow min-cut theorem and in section 4 we prove that there always exist stable outcomes for flow games, arising from convex CMCF-situations.

2. PRELIMINARIES

In the following we consider a directed network with *node set* $P := \{1,2,\ldots,s\}$ and *arc set* $L := \{1,2,\ldots,t\}$. In addition, $N := \{1,2,\ldots,n\}$ denotes the *set of owners (player set) of arcs* and $G := \{1,2,\ldots,m\}$ the *set of commodities*, which are involved in controlled transportations from the *source* $1 \in P$ to the *sink* $s \in P$. The *ownership function* $O : L \to N$ assigns to each arc $\ell \in L$, its controller or owner $O(\ell) \in N$. Finally, $c : L \twoheadrightarrow \mathbb{R}^m_+$ is the *capacity correspondence*, which assigns to each arc ℓ the non-empty subset $c(\ell)$ of the commodity space $\mathbb{R}^m_+$. Here each $u \in c(\ell)$ represents a feasible commodity bundle, which can be transported in one unit of time through arc ℓ, if the owner $O(\ell)$ allows this. Concluding, a *controlled multi-commodity flow situation* (CMCF-situation) is described by the six-tuple

$$\Gamma = \langle P,L,G,c,N,O\rangle$$

The classical *single-commodity* (single owner) *flow situation* (SCF-situation) corresponds to the case $|G| = 1$, $|N| = 1$ and can be denoted

by <P,L,c>.
The *multi-commodity* (single owner) *flow situation* (MCF-situation) can be denoted by <P,L,G,c> and the *single-commodity multi-owner flow situation* (CSCF-situation), as studied in [6], by <P,L,c,N,0>.

Let us call a non-empty subset T of $\mathbb{R}^m_+$ *suitable* if

(S.1) T is compact,

(S.2) T is convex,

(S.3) T is a comprehensive set i.e. $y \in T$ if $0 \le y \le z$ for some $z \in T$.

For many practical situations it is natural to suppose that the capacity set $c(\ell)$ of each arc ℓ is suitable. Condition (S.3) then corresponds to the fact that if a commodity bundle z can pass an arc per unit of time, then also a smaller commodity bundle can pass the arc in such a time interval. Condition (S.2) corresponds to the fact that if the bundles z_1 and z_2 can pass an arc ℓ in the time interval [0,1] and $\alpha \in (0,1)$, then also $\alpha z_1 + (1-\alpha)z_2$ is a suitable commodity bundle e.g. by sending αz_1 through ℓ in time interval $[0,\alpha]$ and sending $(1-\alpha)z_2$ through ℓ in time interval $[\alpha,1]$. Condition (S.1) needs no explanation.

We will call the CMCF-situation (the MCF-situation) *convex* if $c(\ell)$ is a suitable set for each $\ell \in L$. Note that single-commodity flow situations are always convex.

Now we give a sequence of definitions, which play a role in the paper.

(i) A *flow* in Γ (from source 1 to sink s) is a map $f : L \to \mathbb{R}^m_+$ with the following properties:

(F.1) $f(\ell) \in c(\ell)$ for each $\ell \in L$ (*Feasibility property*).

(F.2) $\Sigma\{f(\ell) : \ell \text{ starts in node } p\} = \Sigma\{f(\ell) : \ell \text{ ends in node } p\}$ for each $p \in P\backslash\{1,s\}$ (*Conservation property*).

(F.3) $\Sigma\{f(\ell) : \ell \text{ ends in the source } 1\} = 0$ (*Source property*).

(F.4) $\Sigma\{f(\ell) : \ell \text{ starts in the sink } s\} = 0$ (*Sink property*).

(ii) The *value* v(f) *of a flow* f in Γ is the amount of commodity, entering the sink s per unit of time. Hence,

$$v(f) := \Sigma\{f(\ell) : \ell \text{ ends in the sink } s\}.$$

(iii) The *flow value set* $F(\Gamma)$ of Γ is the set of all possible flow values, thus

$$F(\Gamma) := \{a \in \mathbb{R}^m_+ : a = v(f) \text{ for some flow } f \text{ in } \Gamma\}.$$

(iv) A *cut* of Γ is a subset C of L such that if the arcs in C are deleted, the flow value set of the resulting MCF-situation is equal to $\{0\}$.

(v) The *capacity set of a cut* C is defined by

$$c(C) := \Sigma\{c(\ell) : \ell \in C\}.$$

[Of course, the sum on the right hand side is an algebraic sum of subsets of $\mathbb{R}^m_+$.]

(vi) The *bottle-neck set* $BN(\Gamma)$ of Γ is defined by

$$BN(\Gamma) := \cap\{c(C) : C \text{ is a cut of } \Gamma\}.$$

(vii) Let $S \subset N$ be a non-empty coalition of owners. Then the MCF-*situation, controlled by* S, is the CMCF-situation Γ_S, which we obtain by deleting in the original directed network all those arcs, which do not belong to one of the members of S. Hence, $\Gamma_S = \langle P,L_S,G,c_S,S,O_S\rangle$ and $\Gamma = \Gamma_N$, where $L_S := \{\ell \in L : O(\ell) \in S\}$ and where $O_S : L_S \to S$ and $c_S : L_S \twoheadrightarrow \mathbb{R}^m_+$ are the restrictions to L_S of the maps O and c with domain L.

(viii) The correspondence $V_\Gamma : 2^N \twoheadrightarrow \mathbb{R}^m_+$, which assigns to each non-empty coalition $S \in 2^N$, the flow value set $F(\Gamma_S)$ of the CMCF-situation Γ_S, is called the *flow game arising from* Γ. It is a game, where the payoff set $V_\Gamma(S)$ of a coalition S consists of all those commodity bundles, which can be sent by the coalition S per unit of time from source 1 to sink s, without using arcs in the original network which are owned by agents outside the coalition.

(ix) A *stable outcome of the game* V_Γ is a map $x : N \to \mathbb{R}^m_+$ satisfying

(C.1) $\Sigma_{i\in N}x(i) \in V_\Gamma(N)$,

(C.2) for each coalition S and each $b \in V_\Gamma(S)$ with $b \geq \Sigma_{i\in S}x(i)$, we have $b = \Sigma_{i\in S}x(i)$.

A stable outcome x corresponds to a distribution of the value of a feasible flow for the grand coalition N among its numbers, where owner i is allowed to send a commodity bundle x(i) from the source to the sink per time unit. In view of (C.2) there is no subset of owners, which can make a feasible transportation plan, where each member of that subset is better off. So stable outcomes correspond to stable transportation sharings among the owners and it is interesting to know whether there exist stable outcomes for flow games. In section 4 we prove that convexity of

Γ is a sufficient condition for the existence of stable outcomes for V_Γ.

3. FLOW SITUATIONS

In a single commodity flow situation $\Gamma = \langle P,L,c\rangle$, the sets $c(C)$ for a cut C, $BN(\Gamma)$ and $F(\Gamma)$ are all closed intervals of the form $[0,b] \subset \mathbb{R}$. In our notation, the main results for single-commodity situations read as follows:

(FF.1) the capacity set $c(C) = [0,\gamma(C)]$ of each cut contains the flow value $v(f) \in \mathbb{R}$ of each flow f, so

$$F(\Gamma) \subset BN(\Gamma) \tag{3.1}$$

(FF.2) there exists a (maximal) flow $\hat{f}$ and a (minimal) cut $\hat{C}$ such that $c(\hat{C}) = [0,v(\hat{f})]$, or equivalently:

$$F(\Gamma) = BN(\Gamma). \tag{3.2}$$

What can one say for MCF-situations about the validity of the expressions (3.1) and (3.2), which make also sense for such situations?
Copying a one-commodity proof (cf. [2], lemma 4.1) it is immediately clear that

$$F(\Gamma) \subset BN(\Gamma) \text{ for each MCF-situation } \Gamma. \tag{3.3}$$

Now example 3.1 below shows that (3.2) not necessarily holds for all convex MCF-situations.

<u>Example 3.1.</u> Let $\Gamma = \langle P,L,G,c\rangle$ be the convex MCF-situation with $s = 4$, $t = 5$ and $m = 2$ and with underlying network as drawn in figure 1.

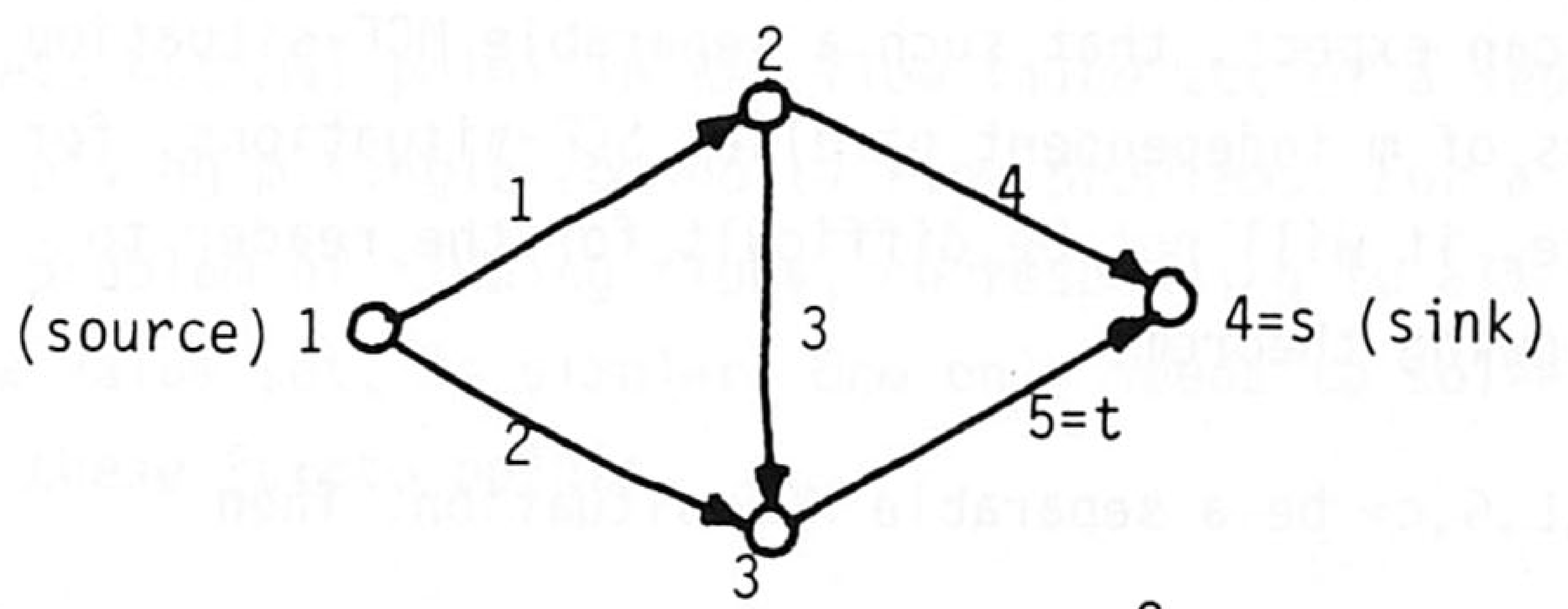

Fig. 1.

Let further, $c(1) = c(3) := \{(u,0) \in \mathbb{R}^2_+ : u \le 4\}$, $c(2) := \{(0,v) \in \mathbb{R}^2_+ : v \le 4\}$, $c(4) := \{(u,v) \in \mathbb{R}^2_+ : u \le 1,\ v \le 4\}$ and $c(5) := \{(u,v) \in \mathbb{R}^2_+ : u+v \le 4\}$. Since each cut of Γ has a subset one of the

cuts $C_1 = \{1,2\}$, $C_2 = \{4,5\}$, $C_3 = \{2,3,4\}$, $C_4 = \{1,5\}$, we obtain:
$BN(\Gamma) = \cap\{c(C_i) : i \in \{1,2,3,4\}\} = c(C_1)$. Thus
$BN(\Gamma) = \{(u,v) \in \mathbb{R}^2_+ : u \le 4,\ v \le 4\}$. We prove that $BN(\Gamma) \ne F(\Gamma)$ by showing that $(4,4) \notin F(\Gamma)$. Suppose, for a moment, that there exists a flow f with $v(f) = (4,4)$. If we denote $f(\ell)$ by (x_ℓ, y_ℓ) then, necessarily: $(x_1,y_1) = (4,0)$, $(x_2,y_2) = (0,4)$. Then $x_5 = x_3 \ge 3$, because $x_4 \le 1$ and $x_3+x_4 = x_1 = 4$, and $y_5 = y_2 = 4$. But then $x_5+y_5 \ge 7 > 4$, which implies that $f(5) = (x_5,y_5) \notin c(5)$ and that is impossible.
Hence, $(4,4) \notin F(\Gamma)$. The example also shows that not necessarily $F(\Gamma) = c(C)$ if C is a cut with $c(C) = BN(\Gamma)$.

Now we introduce two subclasses of convex MCF-situations, for which the bottle-neck sets turn out to coincide with the flow value sets as we see in the theorems 3.3 and 3.4.

Definition 3.2. Let Γ be a MCF-situation and Δ a suitable subset of $\mathbb{R}^m_+$. Then

(i) Γ is called a *separable* MCF-*situation* if for each $\ell \in L$ there is a vector $u(\ell) = (u_1(\ell),\dots,u_m(\ell)) \in \mathbb{R}^m_+$ such that

$$c(\ell) = \{a \in \mathbb{R}^m_+ : 0 \le a \le u(\ell)) = \Pi^m_{g=1}[0,u_g(\ell)].$$

(ii) Γ is called a *uniform* MCF-*situation* with standard set Δ, if for each $\ell \in L$ there is a non-negative number $\delta(\ell)$ (*the diameter of* $c(\ell)$ such that $c(\ell) = \delta(\ell)\Delta$.

In a separable situation the amount of commodity g, which is transportable in a unit time interval through an arc, does not depend on the amounts of the other commodities which are sent through the arcs in the same time interval. One can expect, that such a separable MCF-situation behaves as if it consists of m independent parallel SCF-situations, for each commodity one. Hence, it will not be difficult for the reader to find a proof of the following theorem.

Theorem 3.3. Let $\Gamma = \langle P,L,G,c\rangle$ be a separable MCF-situation. Then $BN(\Gamma) = F(\Gamma)$.

We deal with a uniform MCF-situation if e.g. the arcs consist of pipelines which all have the same possibilities with respect to mixtures of the commodities, but where the diameters of the various pipelines may differ.

Theorem 3.4. Let $\Gamma = \langle P,L,G,c\rangle$ be a uniform MCF-situation with standard set Δ. Then $BN(\Gamma) = F(\Gamma)$.

Proof. Since each cut in Γ has a capacity set of the form $\delta(C)\Delta$, there exists a (smallest) cut C^* with $c(C^*) = \cap\{c(C) : C \text{ is a cut in } \Gamma\} = BN(\Gamma)$. We have to prove that $c(C^*) \subset F(\Gamma)$, knowing by (3.3) that $F(\Gamma) \subset c(C^*)$. Consider the single-commodity flow situation $\Gamma_1 = \langle P,L,\{1\},c^1\rangle$ with capacity correspondence

$$c^1(\ell) := [0,\delta(\ell)] \text{ if } c(\ell) = \delta(\ell)\Delta.$$

Then C^* is a minimal cut in Γ_1 with capacity set $[0,\delta(C^*)]$. By (FF.2) there exists a flow $\hat{f} : L \to \mathbb{R}_+$ in Γ_1 such that $c^1(C^*) = [0,v(\hat{f})]$. Let $a \in \Delta$. Define $\bar{f} : L \to \mathbb{R}^m_+$ by $\bar{f}(\ell) = \hat{f}(\ell)a$. Then $\bar{f}$ is a flow in Γ with $v(\bar{f}) = v(\hat{f})a$.
This implies that $v(\hat{f})\Delta \subset F(\Gamma)$. Since $v(\hat{f})\Delta = \delta(C^*)\Delta = c(C^*)$ we have proved $c(C^*) \subset F(\Gamma)$. Hence, $c(C^*) = F(\Gamma) = BN(\Gamma)$. □

Let us call a flow $f : L \to \mathbb{R}^m_+$ *uniform* if each $f(\ell)$ is a multiple of the same vector. From the above proof the following result is immediate.

Corollary 3.5. For a uniform MCF-situation it is sufficient to consider only uniform flows, because we have:

$$F(\Gamma) = \{v(f) : f \text{ is a uniform flow}\}.$$

We note that one can find a flow with value equal to the unique Pareto optimal point in the flow value set of a separable flow situation by solving m single-commodity flow problems. For a uniform flow situation the problem of finding flows, corresponding to all Pareto points of the flow value set, is simpler. One only needs to solve one SCF-problem for all these Pareto points.

4. FLOW GAMES

First we present a proof of the fact that flow games, which correspond to convex CMCF-situations, possess stable outcomes and then we give an example, showing that convexity is not a superfluous condition.

Theorem 4.1. Let $\Gamma = \langle P,L,G,c,N,O\rangle$ be a convex CMCF-situation. Then there exists a stable outcome.

Proof. We exclude the trivial case $F(\Gamma) = \{0\}$. For $S \in 2^N$, let $e^S \in \mathbb{R}^n$ be the vector with $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ otherwise.

(i) Let $\lambda : 2^N \to [0,\infty)$ be a map with the balancedness property i.e.

$$e^N = \Sigma_{S\in 2^N} \lambda(S)e^S. \tag{4.1}$$

We prove that $\Sigma_{S\in 2^N} \lambda(S)V_\Gamma(S) \subset V_\Gamma(N)$. For each $S \in 2^N$, take $a^S \in V_\Gamma(S)$ and let f^S be a flow in Γ with $f^S(\ell) = 0$ for each ℓ with $O(\ell) \notin S$ and $v(f^S) = a^S$. Then, trivially,

$f := \Sigma_{S\in 2^N} \lambda(S)f^S : L \to \mathbb{R}_+^m$ satisfies the flow properties (F.2), (F.3) and (F.4). Since $f^S(\ell) \in c(\ell)$ for each $\ell \in L$, it follows from $\Sigma_{S\ni O(\ell)} \lambda(S) = 1$ that

$$f(\ell) = \Sigma_{S\in 2^N} \lambda(S)f^S(\ell) = \Sigma_{S\ni O(\ell)} \lambda(S)f^S(\ell) \in \operatorname{conv}(c(\ell)) = c(\ell).$$

So also (F.1) holds. Then f is a flow in Γ with value $\Sigma_{S\in 2^N} \lambda(S)a^S$, which proves that

$$\Sigma_{S\in 2^N} \lambda(S)V_\Gamma(S) \subset V_\Gamma(N). \tag{4.2}$$

(ii) Take an $\hat{a} \in \operatorname{Par}(F(\Gamma))$ for which there is a supporting hyperplane through $\hat{a}$ for the compact and convex set $F(\Gamma)$, which possesses a normal p with only positive coordinates. So we have

$$\langle p,\hat{a}\rangle = \max\{\langle p,a\rangle : a \in F(\Gamma)\},\ p > 0. \tag{4.3}$$

With the aid of the normal p we define the side payment game $w : 2^N \to \mathbb{R}_+$ by

$$w(S) := \max\{\langle p,a\rangle : a \in V_\Gamma(S)\} \text{ for each } S \in 2^N. \tag{4.4}$$

For each $S \in 2^N$, take $b^S \in V_\Gamma(S)$ with $w(S) = \langle p,b^S\rangle$.

From (4.2) we conclude that $\Sigma_{S\in 2^N} \lambda(S)b^S \in V_\Gamma(N)$ if $\lambda : 2^N \to [0,\infty)$ satisfies (4.1). This implies that

$$w(N) \geq \langle p,\Sigma_{S\in 2^N} \lambda(S)b^S\rangle = \Sigma_{S\in 2^N} \lambda(S)w(S).$$

Hence, w is a balanced side payment game. By the Bondareva-Shapley theorem (cf. [7], p.157), we may conclude that w possesses a core element i.e. there exists a function $\xi : N \to \mathbb{R}_+$ with

$$\Sigma_{i\in N}\ \xi(i) = w(N),\ \Sigma_{i\in S}\ \xi(i) \geq w(S) \text{ for each } S \in 2^N. \qquad (4.5)$$

(iii) We now define $x : N \to \mathbb{R}^m_+$ by $x(i) := \xi(i)(w(N))^{-1}\hat{a}$ for each $i \in N$ and show that x is a stable outcome. By (4.5), $\Sigma_{i\in N}\ x(i) = \hat{a}$, so x has property (C.1). To prove (C.2), we take $S \in 2^N$, $y \in \mathbb{R}^m_+$ with $y \geq \Sigma_{i\in S}\ x(i)$, $y \neq \Sigma_{i\in S}\ x(i)$.
Because $p > 0$, we obtain in view of (4.3), (4.4) and (4.5):

$$\langle p,y\rangle > \Sigma_{i\in S} \langle p,x(i)\rangle = \Sigma_{i\in S}\ \xi(i) \geq w(S).$$

Then, by (4.4), $y \notin V_\Gamma(S)$. This proves (C.2). □

Example 4.2. Let $\Gamma = \langle P,L,G,c,N,O\rangle$ be the CMCF-situation with s = 3, t = 6, m = 2 and n = 3 and with underlying network as in figure 2.

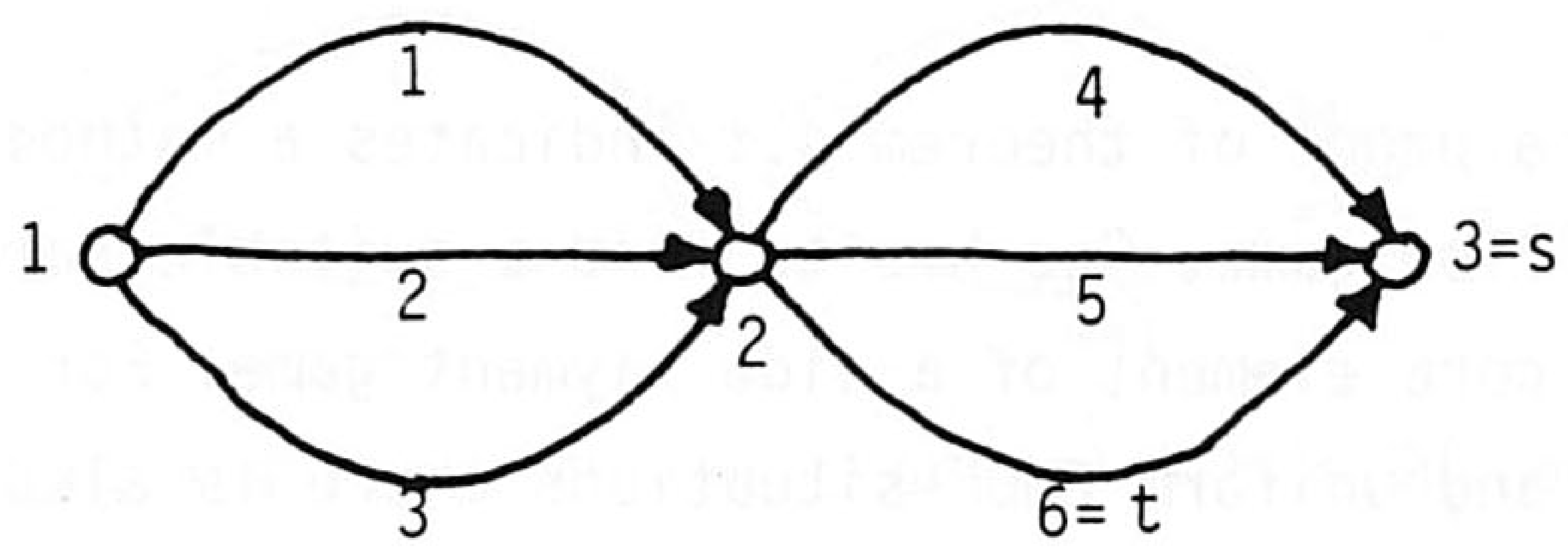

Fig.2.

Let further $O(1) = O(4) = 1$, $O(2) = O(5) = 2$, $O(3) = O(6) = 3$, and let
$c(\ell) := \{(u,v) \in \mathbb{R}^2_+ : u \leq 1,\ v \leq 1\}$ if $\ell \in \{1,2,3\}$,
$c(\ell) := \{(u,0) \in \mathbb{R}^2_+ : u \leq 2\} \cup \{(0,v) \in \mathbb{R}^2_+,\ v \leq 2\}$ if $\ell \in \{4,5,6\}$.
Then c(4) is not convex. For the corresponding game we have for each $i \in N$:

$$V_\Gamma(\{i\}) = \{(u,0) \in \mathbb{R}^2_+ : u \leq 1\} \cup \{(0,v) \in \mathbb{R}^2_+ : v \leq 1\}$$
$$V_\Gamma(N\backslash\{i\}) = \{(u,v) \in \mathbb{R}^2_+ : u \leq 2,\ v \leq 2\} \qquad (4.6)$$
$$V_\Gamma(N) = \{(u,v) \in \mathbb{R}^2_+ : u \leq 2,\ v \leq 3\} \cup \{(u,v) \in \mathbb{R}^2_+ : u \leq 3,\ v \leq 2\}.$$

We prove that stable outcomes of this game do not exist. Suppose, contrarily, that x is a stable outcome. Since $\mathrm{Par}(V_\Gamma(N)) = \{(3,2),(2,3)\}$ we can suppose w.l.o.g. that

$$x(1) + x(2) + x(3) = (3,2). \qquad (4.7)$$

Let $z(1) := x(2)+x(3)$, $z(2) := x(1)+x(3)$, $z(3) := x(2)+x(1)$. Then from

(4.7) we obtain

$$z(1)+z(2)+z(3) = (6,4) \tag{4.8}$$

and from (4.6) it follows that $\{(2,2)\} = \text{Par } V_\Gamma(N\backslash\{i\})$, so

$$z(i) = (2,2) \text{ or at least one coordinate is greater than 2.} \tag{4.9}$$

Using (4.8) and (4.9) one can easily prove that $z(i) \neq (2,2)$ for each $i \in N$. Then w.l.o.g. there exist $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\delta_1 \geq 0$, $\delta_2 \geq 0$ such that

$$z(1) = (2+\varepsilon_1,\delta_1),\ z(2) = (2+\varepsilon_2,\delta_2),\ z(3) = (2-\varepsilon_1-\varepsilon_2,4-\delta_1-\delta_2) \tag{4.10}$$

Then

$$x(1) = (1-\varepsilon_1,2-\delta_1),\ x(2) = (1-\varepsilon_2,2-\delta_2),\ x(3) = (1+\varepsilon_1+\varepsilon_2,-2+\delta_1+\delta_1). \tag{4.11}$$

Since the first coordinate of $z(3)$ is smaller than 2, by (4.9) we have $4-(\delta_1+\delta_2) > 2$. But then it follows from (4.11) that the second coordinate of $x(3)$ is smaller than zero and that is in contradiction with the assumption that x is a stable outcome.

In principle, the proof of theorem 4.1 indicates a method to find a stable outcome for a flow game. One has to find a suitable Pareto-optimal point of $V_\Gamma(N)$ and a core element of a side payment game. For the sub-classes of separable and uniform CMCF-situations there is also an easy method to find stable outcomes, based on the idea of Kalai and Zemel [6] for single-commodity games.
We describe this in the following theorems, the proof of these theorems we leave to the reader.

Theorem 4.3. Let $\Gamma_N = \langle P,L,G,c,N,O\rangle$ be a uniform CMCF-situation. Let $a \in \text{Par } F(\Gamma_N)$. Let C^* be the minimal cut and $\hat{f}$ the flow as in the proof of theorem 3.4. Let for each $i \in N$:

$$\xi(i) := \Sigma\{\hat{f}(\ell) : \ell \in C^*, O(\ell) = i\}.$$

Then $x : N \to \mathbb{R}^m_+$, defined by $x(i) := (v(\hat{f}))^{-1}\xi(i)a$ is a stable outcome of the flow game arising from Γ_N with $\Sigma_{i\in N}\, x(i) = a$.

Let $\Gamma = \langle P,L,G,c\rangle$ be a separable MCF-situation and let $f : L \to \mathbb{R}^m_+$ be a flow in Γ with value the unique Pareto optimal point in the flow value set of Γ. In view of (FF.2) and theorem 3.3 it is obvious that for each

$g \in G$ there exists a cut C_g with property

$$\{x \in \mathbb{R} : xe^{\{g\}} \in c(C_g)\} = [0, v_g(f)].$$

We now can state the next theorem.

Theorem 4.4. Let Γ_N be a separable CMCF-situation. Let f and C_g be as above. For $g \in G$, let

$$x_g(i) := \Sigma\{f_g(\ell) : \ell \in C_g,\ O(\ell) = i\}.$$

Then $x : N \to \mathbb{R}^m_+$ with $x(i) := (x_1(i), \ldots, x_g(i), \ldots, x_m(i))$ is a stable outcome of the flow game corresponding to Γ_N.

The following example may be illustrative.

Example 4.5. Let $\Gamma^1 = \langle P,L,G,c^1,N,O\rangle$ and $\Gamma^2 = \langle P,L,G,c^2,N,O\rangle$ be CMCF-situations with s = 3, t = 4, m = 2, n = 4 and underlying network as in Figure 3.

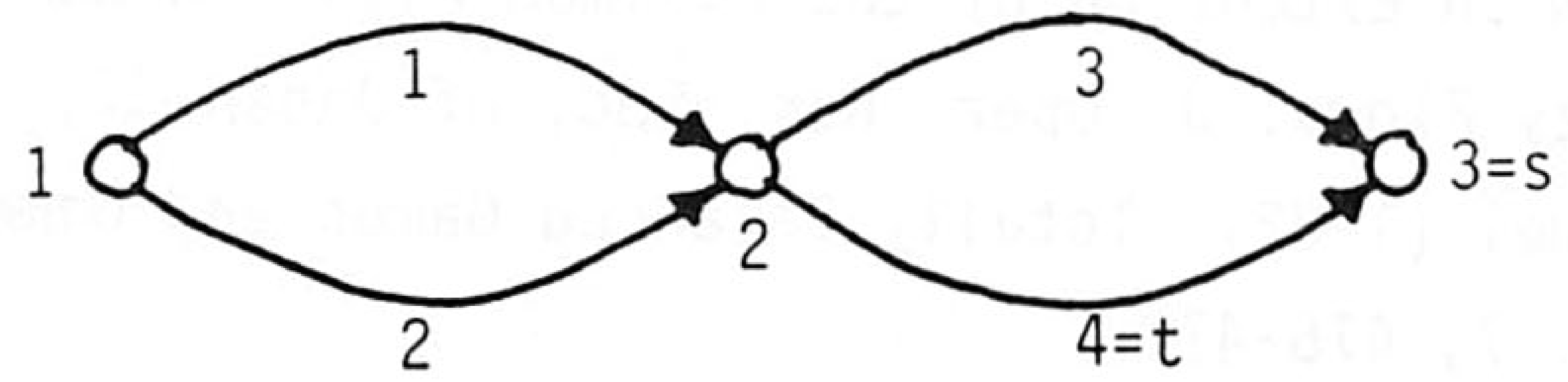

Fig.3.

Let, further, $O(\ell) = \ell$ for each $\ell \in L$ and $c^1 : L \to \mathbb{R}^2_+$ and $c^2 : L \to \mathbb{R}^2_+$ be given by

$$c^1(1) := [0,1] \times [0,3],\ c^1(2) := [0,2] \times [0,6],$$
$$c^1(3) := [0,3] \times [0,2],\ c^1(4) := [0,5] \times [0,4].$$

and $c^2(\ell) := \ell\Delta$ for all $\ell \in \{1,2,3,4\}$, where $\Delta := \{(u,v) \in \mathbb{R}^2_+ : u^2+v^2 \leq 1\}$. Then Γ^1 is a separable CMCF-situation, $C_1 = \{1,2\}$, $x_1(1) = 1$, $x_1(2) = 2$, $C_2 = \{3,4\}$, $x_2(3) = 2$, $x_2(4) = 4$. So $x : N \to \mathbb{R}^2_+$, with $x(1) = (1,0)$, $x(2) = (2,0)$, $x(3) = (0,2)$, $x(4) = (0,4)$ is a stable outcome of the game V_{Γ^1}.

The situation Γ^2 is a uniform one with minimum cut $C^* = \{1,2\}$. Then each $x : N \to \mathbb{R}^2_+$ of the form $x(1) = a$, $x(2) = 2a$, $x(3) = x(4) = 0$ with $a_1^2+a_2^2 = 1$ is a stable outcome.

Since each subgame of a multi-commodity flow game is of the same type, we have also proved that flow games, which correspond to convex CMCF-situations are totally balanced games. We do not touch here the question whether, conversely, each totally balanced game $V : 2^N \twoheadrightarrow \mathbb{R}^m_+$ with $V(S)$ suitable for each S, can be obtained from a convex CMCF-situation. But we finish with the remark that Kalai and Zemel [6] proved that all totally balanced side payment games with non-negative worths, can be generated by CSCF-situations.

References

[1] L.R. Ford, D.R. Fulkerson (1956): Maximal Flow Through a Network. Canad. J. Math. 8, 399-404.

[2] L.R. Ford, D.R. Fulkerson (1962): Flows in Networks. Princeton University Press, Princeton, NJ.

[3] T.C. Hu (1963): Multi-Commodity Network Flows. Oper. Res. 11, 344-360.

[4] T.C. Hu (1969): Integer Programming and Network Flows. Addison-Wesley, Reading, Mass.

[5] M. Iri (1971): On an Extension of the Maximum-Flow Minimum-Cut Theorem to Multicommodity Flows. J. Oper. Res. Soc. of Japan 13, 129-135.

[6] E. Kalai, E. Zemel (1982): Totally Balanced Games and Games of Flow. Math. Oper. Res. 7, 476-478.

[7] G. Owen (1982): Game Theory. Academic Press, New York.

[8] B. Rothschild, A. Whinston (1966): On Two-Commodity Network Flows. Oper. Res. 14, 377-387.

[9] A. Schrijver (1983): Min-Max Results in Combinatorial Optimization. In: Mathematical Programming. The State of the Art (Eds. A. Bachem, M. Grötschel, B. Korte). Springer Verlag, Berlin, pp. 439-500.

[10] P.D. Seymour (1977): The matroids with the max-flow min-cut property. J. Combinatorial Theory (B) 23, 189-222.

[11] P.D. Seymour (1979): A short proof of the two-commodity flow theorem. J. Combinatorial Theory (B) 26, 370-371.

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